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On Coulomb wave functions

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Introduction

The Appell functions [1], which are generalizations of the hypergeometric functions to functions of two variables, have been used in the quantum mechanical treatment of problems connected with the motion of charged particles in a Coulomb field. An extensive application is found in the study of nuclear structure by Coulomb excitation presented recently by K. Alder, A. Bohr, T. Huus, B. Mottelson and A. Winther [2].

From the fact that the matrix elements for electromagnetic transitions between two states in an external Coulomb field can be expressed in terms of Appell functions in various ways and that these functions in many cases furnish analytic continuations of each other, their usefulness is obvious. These matrix elements may be looked upon as the Laplace transforms of radial Coulomb wave functions or products of such functions, and the many relations between different Appell functions allow a variety of expressions for the transition matrix elements. On the other hand the Laplace transforms of confluent Appell functions, when integrated over one of the variables only or with a simple relation between the two variables in order to form a function of one variable, often leads to expressions that are very similar to those of the transition matrix elements. This indicates a connection between radial Coulomb wave functions and some confluent Appell function. Since the Coulomb wave functions for non-bound states are rather cumbersome to work with it was considered worth while to look for a representation of the radial Coulomb wave functions in terms of a confluent Appell function. There were reasons to believe that the latter function, in some respects at least, should have more attractive features than the Whittaker functions ordinarily used in this connection. An investigation based on these arguments is presented below.

Real valued wave functions have been chosen throughout. This is not a necessity, but the symmetry gained thereby seems to make it desirable.

Although the transition matrix elements between two Coulomb states are conveniently expressible in Appell functions there remains the difficulty that the power series expansions of the Appell functions, which perhaps provide the simplest way of obtaining numerical values except possibly for very special cases, often do not converge or converge very poorly for practical purposes.

The functions introduced in the present paper allow more transformations than the ordinary Coulomb wave functions to which they are reduced for special values of the parameters and seem to offer more possibilities to extend the domain of convergence.

Radial Coulomb wave functions

We have chosen to treat the relativistic case but there is no difficulty in translating the derived formulas to their non-relativistic analogies. Since the angular dependence of a partial wave is the same for all potentials with spherical symmetry and this angular dependence may be found elsewhere [3] we shall be concerned here with the radial functions only. They satisfy the differential equations [3]

$$\begin{aligned}\frac{df(r)}{dr}&=\frac{\varkappa-1}{r}f(r)-\left(E-1+\frac{\alpha Z}{r}\right)g(r)\\ \frac{dg(r)}{dr}&=\left(E+1+\frac{\alpha Z}{r}\right)f(r)-\frac{\varkappa+1}{r}g(r),\end{aligned}\tag{1}$$

where E is the energy measured in units of mc^2, r is measured in units of $\hbar/mc$ and $\varkappa$ is connected with the total angular momentum number by the relation $\pm\varkappa=j+1/2$.

It is convenient to introduce new variables

$$\begin{aligned}\chi_1(r)&=rf(r)\\ \chi_2(r)&=rg(r)\end{aligned}\tag{2}$$

and sometimes also a new independent variable $t=2\alpha Zr$, which leads to the equations

$$\begin{aligned}\frac{d\chi_1}{dt}-\frac{\varkappa}{t}\chi_1+\left(\frac{E-1}{2\alpha Z}+\frac{\alpha Z}{t}\right)\chi_2&=0\\ \frac{d\chi_2}{dt}-\left(\frac{E+1}{2\alpha Z}+\frac{\alpha Z}{t}\right)\chi_1+\frac{\varkappa}{t}\chi_2&=0\end{aligned}\tag{3}$$

We claim that the equations (3) have solutions of the form

$$\begin{aligned}\chi_1(t)&=\frac{\alpha Z}{\varkappa-\gamma}t^{\gamma}\left[\frac{E-1}{2}\Phi(\gamma,E,t)+(\gamma-\varkappa)\frac{d\Phi(\gamma,E,t)}{dt}\right]\\ \chi_2(t)&=-t^{\gamma}\left[\frac{E+1}{2}\Phi(\gamma,E,t)+(\gamma+\varkappa)\frac{d\Phi(\gamma,E,t)}{dt}\right],\end{aligned}\tag{4}$$

where $$\gamma=\pm\sqrt{\varkappa^2-\alpha^2Z^2}$$

and $\Phi(\gamma,E,t)$ is any solution of

$$\frac{d^2\Phi(\gamma,E,t)}{dt^2}+\frac{2\gamma}{t}\frac{d\Phi(\gamma,E,t)}{dt}+\left(\frac{E}{t}+\frac{E^2-1}{4\alpha^2Z^2}\right)\Phi(\gamma,E,t)=0.\tag{5}$$

This is easily verified if the expressions (4) are introduced into the equations (3). Both expressions on the left hand side of (3) are then found to be identically zero if $\Phi(\gamma,E,t)$ satisfies (5). This equation has, independently of the sign of γ, always a solution that is finite for $t=0$ and which is uniquely determined by its value at this point. We shall denote by $\Phi(\gamma, E, t)$ the solution of (5) for which

$$\Phi(\gamma, E, 0) = 1.$$

It follows that two independent solutions of the equations (3) are obtained from (4), corresponding to the two possible signs of γ, and the problem of solving (3) is reduced to finding the regular solution $\Phi(\gamma, E, t)$ of (5).

The solutions $\Phi(\gamma, E, t)$ of the equation (5) are well known and are exponential functions times confluent hypergeometric functions. We wish, however, to point out that the regular solution of (5) may be conveniently represented by a confluent Appell function [1]. The function in question is

$$\Phi_2(\beta, \beta'; 2\gamma; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(\beta)_m (\beta')_n x^m y^n}{(2\gamma)_{m+n}\, m!\, n!}. \tag{6}$$

The double series is convergent for all values of x and y independently of the values of the parameters with the exception of $2\gamma =$ negative integer or zero. This exception is obvious from the definition of the symbols $(\beta)_m$ etc. which are abbreviations for products of the type

$$(\beta)_m = \beta(\beta+1)(\beta+2)\ldots(\beta+m-1).$$

In order to obtain a function of one variable we put $x = qt$ and $y = st$:

$$\Phi_2(\beta, \beta'; 2\gamma; qt, st) = \sum_{k=0}^{\infty} \frac{t^k}{(2\gamma)_k} \sum_{m+n=k} \frac{(\beta)_m (\beta')_n q^m s^n}{m!\, n!}.$$

The second sum may be expressed in a hypergeometric function: For [4]

$$(1-qt)^{-\beta}(1-st)^{-\beta'} = \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} (\beta+\beta')_k\, {}_2F_1\left(-k; \beta'; \beta+\beta'; 1-\frac{s}{q}\right)$$

where

$${}_2F_1\left(-k; \beta'; \beta+\beta'; 1-\frac{s}{q}\right) = \sum_{n=0}^{k} \frac{(-k)_n (\beta')_n}{(\beta+\beta')_n} \frac{\left(1-\frac{s}{q}\right)^n}{n!}.$$

Also more directly

$$(1-qt)^{-\beta}(1-st)^{-\beta'} = \sum_{k=0}^{\infty} t^k \sum_{m+n=k} \frac{(\beta)_m (\beta')_n q^m s^n}{m!\, n!}.$$

Hence

$$\sum_{m+n=k} \frac{(\beta)_m (\beta')_n q^m s^n}{m!\, n!} = \frac{(\beta+\beta')_k q^k}{k!}\, {}_2F_1\left(-k; \beta'; \beta+\beta'; 1-\frac{s}{q}\right)$$

and consequently

$$\Phi_2(\beta, \beta'; 2\gamma; qt, st) = \sum_{n=0}^{\infty} \frac{(\beta+\beta')_n (qt)^n}{(2\gamma)_n\, n!}\, {}_2F_1\left(-n; \beta'; \beta+\beta'; 1-\frac{s}{q}\right). \tag{7}$$

To get a solution of (5) we make the following choice for the parameters

$$\beta=\gamma+i\eta,\ \beta'=\gamma+i\eta,\ q=-s=\frac{ip}{2\alpha Z} \tag{8}$$

where

$$\eta=\frac{E\alpha Z}{p},\ p=\sqrt{E^2-1}$$

and obtain

$$\Phi_2(\gamma+i\eta,\gamma-i\eta;2\gamma;ipr,-ipr)=\sum_{n=0}^{\infty}\frac{(ipr)}{n!}\,{}_2F_1(-n,\gamma-i\eta;2\gamma;2) \tag{9}$$

This function is real valued and solves equation (5). The first statement follows directly from the definition of the function and the second statement can be proved as follows.

Put

$$\Phi(\gamma,E,t)=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!} \tag{10}$$

and use equation (5) to determine the coefficients c_n. The following recursion formula will be obtained

$$(n+2\gamma)\,c_{n+1}+E\,c_n+\frac{p^2}{4\alpha^2Z^2}\,n\,c_{n-1}=0,$$

which is satisfied by the real coefficients

$$c_n=\left(\frac{ip}{2\alpha Z}\right)^n{}_2F_1(-n;\gamma-i\eta;2\gamma;2) \tag{11}$$

since from the recursion formulas for the hypergeometric functions it can be shown that

$$(n+2\gamma)\,{}_2F_1(-n-1,\gamma-i\eta;2\gamma;2)-2i\eta\,{}_2F_1(-n,\gamma-i\eta;2\gamma;2)-$$
$$-n\,{}_2F_1(-n+1,\gamma-i\eta;2\gamma;2)=0.$$

Then

$$\Phi(\gamma,E,t)=\sum_{n=0}^{\infty}\frac{(ipr)^n}{n!}\,{}_2F_1(-n,\gamma-i\eta;2\gamma;2)$$

which is the function (9) and thus

$$\Phi(\gamma,E,t)=\Phi_2(\gamma+i\eta,\gamma-i\eta;2\gamma;ipr,-ipr) \tag{12}$$

The series expansion (10) for $\Phi(\gamma,E,t)$ is convergent for all values of t and E as a consequence of the afore-mentioned convergence of the Φ_2-function of which it is a special case. The coefficients c_n are polynomials in E, as can be seen at once from their recursion formula, and therefore $\Phi(\gamma,E,t)$ is an analytic function of both the variables E and t for any finite value of these variables. There is no need then to distinguish between bound and non-bound states from a formal point of view.

The function (12) is real and ip enters only in even powers. As mentioned before it supplies us with two independent solutions of the Dirac equation through (4). The solution that is regular at the origin is obtained for $\gamma = +\sqrt{\varkappa^2 - \alpha^2 Z^2}$ and the irregular solution for $\gamma = -\sqrt{\varkappa^2 - \alpha^2 Z^2}$.

It has been mentioned already that $\Phi(\gamma, E, t)$ must be an exponential function times a confluent hypergeometric function and it is not difficult to show that

$$\Phi(\gamma, E, t) = e^{-ipr}\,{}_1F_1(\gamma + i\eta; 2\gamma; 2ipr), \tag{13}$$

e.g. by solving (5). We will, however, prove (13) in a different manner and at the same time arrive at a more general result, that will be useful in applications. The notation ${}_pF_q(\alpha_1, \alpha_2, \dots \alpha_p; \beta_1, \beta_2, \dots \beta_q; x)$ will be used throughout this paper for a generalized hypergeometric function. They are defined by the power series expansion

$${}_pF_q(\alpha_1, \alpha_2, \dots \alpha_p; \beta_1, \beta_2, \dots \beta_q; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_q)_n} \frac{x^n}{n!}$$

and its analytic continuations.

The proof of (13) is based on the Appell function [1]

$$F_1(\alpha, \beta, \beta'; 2\gamma; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n x^m y^n}{(2\gamma)_{m+n}\, m!\, n!} \tag{14}$$

from which the confluent Appell function Φ_2 is obtained by the limiting process of confluence:

$$\Phi_2(\beta, \beta'; 2\gamma; x, y) = \lim_{\varepsilon \to 0} F_1\left(\frac{1}{\varepsilon}, \beta, \beta'; 2\gamma; \varepsilon x, \varepsilon y\right) \tag{15}$$

as

$$\lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon}\right)_{m+n} (\varepsilon x)^m (\varepsilon y)^n = x^m y^n \cdot$$

Before computing the limit in (15) a well known transformation of the F_1-function is applied [1]:

$$F_1(\alpha, \beta, \beta'; 2\gamma; x, y) = (1-x)^{-\alpha} F_1\left(\alpha, 2\gamma - \beta - \beta', \beta'; 2\gamma; \frac{x}{x-1}, \frac{y-x}{1-x}\right) \cdot$$

Then

$$\Phi_2(\beta, \beta'; 2\gamma; x, y) =$$

$$= \lim_{\varepsilon \to 0} (1 - \varepsilon x)^{-1/\varepsilon} F_1\left(\frac{1}{\varepsilon}, 2\gamma - \beta - \beta', \beta'; 2\gamma; \frac{\varepsilon x}{\varepsilon x - 1}, \frac{\varepsilon (y-x)}{1 - \varepsilon x}\right)$$

$$= e^x \Phi_2(2\gamma - \beta - \beta'; \beta'; 2\gamma; -x, y - x) \tag{16}$$

and if we put $\beta' + \beta = 2\gamma$ in agreement with (8) the Φ_2-function on the right hand side is reduced to a confluent hypergeometric function

$$\Phi_2(0, \beta'; 2\gamma; x, y-x) = {}_1F_1(\beta'; 2\gamma; y-x)$$

This is a direct consequence of the definition (6) of the Φ_2-function and proves that the Φ_2-function is reduced to an exponential function times a confluent hypergeometric function if $\beta+\beta'=2\gamma$ or

$$\Phi_2(\beta, \beta'; \beta+\beta'; x, y) = e^x\, {}_1F_1(\beta'; \beta+\beta'; y-x). \tag{17}$$

Also for any λ

$$e^\lambda \Phi_2(\beta, \beta'; \beta+\beta'; x, y) = \Phi_2(\beta, \beta'; \beta+\beta'; x+\lambda; y+\lambda).$$

Using the notations (8) we obtain

$$\Phi_2(\gamma+i\eta, \gamma-i\eta; 2\gamma; ipr, -ipr) = e^{ipr}\, {}_1F_1(\gamma-i\eta; 2\gamma; -2ipr) =$$
$$= e^{-ipr}\, {}_1F_1(\gamma+i\eta; 2\gamma; 2ipr),$$

where the last equality follows from the fact that the function is real valued. This proves (13) and connects our formalism with more familiar expressions for the Coulomb wave functions. These functions are usually given in terms of the function

$$H(r) = e^{-ipr}\, {}_1F_1(\gamma+1+i\eta; 2\gamma+1; 2ipr)$$

This function may be expressed in terms of the functions $\Phi(\gamma, E, t)$ with the aid of recursion relations for the ${}_1F_1$-functions:

$$Re\, H = \tfrac{1}{2}(H+H^*) = \tfrac{1}{2} e^{-ipr}\, [{}_1F_1(1+\gamma+i\eta; 1+2\gamma; 2ipr) +$$
$$+ {}_1F_1(\gamma+i\eta; 1+2\gamma; 2ipr] = \tfrac{1}{2} e^{-ipr} \Big[{}_1F_1(\gamma+i\eta; 1+2\gamma; 2ipr) +$$
$$+ \frac{2ipr}{1+2\gamma}\, {}_1F_1(1+\gamma+i\eta; 2\gamma+2; 2ipr) + {}_1F_1(\gamma+i\eta; 1+2\gamma; 2ipr \Big]$$
$$= e^{-ipr} \left[{}_1F_1(\gamma+i\eta; 1+2\gamma; 2ipr) + \frac{2ipr}{2(1+2\gamma)}\, {}_1F_1(1+\gamma+i\eta; 2\gamma+2; 2ipr \right]$$
$$= e^{-ipr} \left[{}_1F_1(\gamma+i\eta; 2\gamma; 2ipr) + \frac{Et}{(2\gamma)_2}\, {}_1F_1(1+\gamma+i\eta; 2\gamma+2; 2ipr) \right]$$
$$= \Phi(\gamma, E, t) + \frac{Et}{(2\gamma)_2} \Phi(\gamma+1, E, t), \tag{18}$$

and for the imaginary part after similar calculations

$$Im\, H = \frac{pr}{2(1+2\gamma)} \Phi(\gamma+1, E, t). \tag{19}$$

Since generally

$$\frac{d\, {}_1F_1(a; c; x)}{dx} = \frac{a}{c}\, {}_1F_1(a+1; c+1; x)$$

it follows that

$$\frac{d\Phi(\gamma,E,t)}{dt} = -\frac{ip}{2\alpha Z}\Phi(\gamma,E,t) + \left(\frac{ip}{2\alpha Z} - \frac{E}{2\gamma}\right) H$$

and since $\frac{d\Phi(\gamma,E,t)}{dt}$ is real valued

$$\frac{d\Phi(\gamma,E,t)}{dt} = -\frac{p}{2\alpha Z} Im H - \frac{E}{2\gamma} Re H$$

and with the aid of (18) and (19) finally

$$-2\gamma\frac{d\Phi(\gamma,E,t)}{dt} = E\,\Phi(\gamma,E,t) + t\frac{E^2+\frac{p^2\gamma^2}{\alpha^2 Z^2}}{(2\gamma)_2}\Phi(\gamma+1,E,t). \tag{20}$$

It is sometimes an advantage not to have the derivative of $\Phi(\gamma,E,t)$ in the wave functions. With the aid of (20) it can be eliminated yielding

$$\begin{aligned} \chi_1 &= \frac{\alpha Z}{\varkappa-\gamma}\frac{E\varkappa-\gamma}{2\gamma}t^{\gamma}\left[\Phi(\gamma,E,t) + \frac{E\varkappa+\gamma}{\varkappa+\gamma}\frac{t}{(2\gamma)_2}\Phi(\gamma+1,E,t)\right] \\ \chi_2 &= \frac{E\varkappa-\gamma}{2\gamma}t^{\gamma}\left[\Phi(\gamma,E,t) + \frac{E\varkappa+\gamma}{\varkappa-\gamma}\frac{t}{(2\gamma)_2}\Phi(\gamma+1,E,t)\right] \end{aligned} \tag{21}$$

For the sake of completeness we will also derive the normalization constant for the wave functions for non-bound states. The normalization constant for bound states is found e.g. in reference [3], and correcting it to fit the solutions (21) is a very simple operation. The solutions are normalized to a flux of one particle per unit energy interval per second [5]

$$\begin{aligned} N_f\chi_1 &\approx \sqrt{\frac{E-1}{\pi p}}\sin\left[pr+\eta\ln 2pr-\frac{\pi\gamma}{2}-\arg\Gamma(\gamma+i\eta)+\delta\right] \\ N_f\chi_2 &\approx -\sqrt{\frac{E+1}{\pi p}}\cos\left[pr+\eta\ln 2pr-\frac{\pi\gamma}{2}-\arg\Gamma(\gamma+i\eta)+\delta\right] \end{aligned} \tag{22}$$

where δ is an asymptotic phase. This asymtotic form is most easily obtained from the well known asymtotic behavior of the function ${}_1F_1(\gamma+i\eta;2\gamma;2ipr)$ for large arguments.

$${}_1F_1(\gamma+i\eta;2\gamma;2ipr) \approx \Gamma(2\gamma)\left[\frac{e^{i\pi\gamma}e^{-\pi\eta}}{\Gamma(\gamma-i\eta)(2ipr)^{\gamma+i\eta}} + \frac{e^{2ipr}}{\Gamma(\gamma+i\eta)(2ipr)^{\gamma-i\eta}}\right]$$

Using (13) we obtain

$$\Phi(\gamma,E,t) \approx \frac{2\Gamma(2\gamma)e^{-\frac{\pi\eta}{2}}}{|\Gamma(\gamma+i\eta)|(2ipr)^{\gamma}}\cos\left[pr+\eta\ln 2pr-\frac{\pi\gamma}{2}-\arg\Gamma(\gamma+i\eta)\right]$$

$$\frac{d\Phi(\gamma,E,t)}{dt}\approx-\frac{p}{\alpha Z}\frac{\Gamma(2\gamma)\,e^{-\frac{\pi\eta}{2}}}{|\Gamma(\gamma+i\eta)|\,(2ipr)^{\gamma}}\sin\left[pr+\eta\ln 2pr-\frac{\pi\gamma}{2}-\arg\Gamma(\gamma+i\eta)\right]\cdot$$

These expressions combined with (4) give the asymptotic form (22) where

$$e^{-2i\delta}=-\frac{\gamma+i\eta}{\varkappa-\dfrac{i\alpha Z}{p}}$$

or

$$\operatorname{tg}\delta=\frac{\alpha Z(E-1)}{p(\varkappa-\gamma)}\tag{23}$$

and

$$N_f=\left(\frac{p}{\alpha Z}\right)^{\gamma}\sqrt{\frac{\varkappa-\gamma}{E\varkappa-\gamma}}\,\frac{|\Gamma(\gamma+i\eta)|\,e^{\frac{\pi\eta}{2}}}{\sqrt{2\pi p}\,\Gamma(2\gamma)}.\tag{24}$$

The second expression for the asymptotic phase is a consequence of the first formula. In the real valued formalism presented here the first expression never will be needed but it is the definition of δ usually encountered in papers dealing with Coulomb wave functions. Writing down the normalized wave functions we have

$$N_f\chi_1=\alpha Z N t^{\gamma}\left[\Phi(\gamma,E,t)+\frac{E\varkappa+\gamma}{\varkappa+\gamma}\frac{t}{(2\gamma)_2}\Phi(\gamma+1,E,t)\right]$$

$$N_f\chi_2=(\varkappa-\gamma)N t^{\gamma}\left[\Phi(\gamma,E,t)+\frac{E\varkappa+\gamma}{\varkappa-\gamma}\frac{t}{(2\gamma)_2}\Phi(\gamma+1,E,t)\right]\tag{25}$$

where

$$N=\left(\frac{p}{\alpha Z}\right)^{\gamma}\sqrt{\frac{E\varkappa-\gamma}{\varkappa-\gamma}}\,\frac{|\Gamma(\gamma+i\eta)|\,e^{\frac{\pi\eta}{2}}}{\sqrt{2\pi p}\,\Gamma(2\gamma+1)}.$$

In deriving (25) care has to be taken not to loose a change of sign which shall be there when the quantum number $\varkappa$ changes its sign.

The quantity under the square root sign is positive for all occurring values of the three parameters involved. It replaces in a certain sense the factors $\sqrt{1-E}$ and $\sqrt{1+E}$ in the ordinary expression for the relativistic Coulomb wave functions, but here the change from bound to non-bound states introduces no imaginary quantities. It should perhaps be remarked that the normalized irregular wave functions may still be obtained by changing the sign of γ wherever it occurs.

Integral representations

Integral representations for the Φ_2-function valid for any values of the parameters are possible only as contour integrals in the complex plane. Such a representation has been given by Erdélyi [6] and a representation in the form of a double integral over a real domain by Appell and Kampé de Fériet [1]. Integral representations are useful tools in the transformation theory of the function represented and also in connection with analytic continuations.

We will show that the Φ_2-function has an integral representation

$$\Phi_2(\beta, \beta'; 2\gamma; x, y) = \\ = \frac{\Gamma(2\gamma)}{\Gamma(\beta)\Gamma(2\gamma-\beta)} \int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma-\beta-1} {}_1F_1[\beta'; 2\gamma-\beta; y(1-u)]\, du \tag{26}$$

The integral has a meaning only when $Re\, 2\gamma > Re\, \beta > 0$ but results derived from it are in most cases free from this restriction. To prove (26) we write

$$\int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma-\beta-1} {}_1F_1(\beta'; 2\gamma-\beta; y(1-u))\, du =$$

$$= \int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\beta')_n y^n (1-u)^n}{(2\gamma-\beta)_n\, n!}\, du =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta')_n y^n}{(2\gamma-\beta)_n\, n!} \int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma+n-\beta-1}\, du =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta')_n y^n}{(2\gamma-\beta)_n\, n!} \frac{\Gamma(\beta)\Gamma(2\gamma+n-\beta)}{\Gamma(2\gamma+n)} {}_1F_1(\beta; 2\gamma+n; x)$$

as
$$ {}_1F_1(\beta; 2\gamma+n; x) = \frac{\Gamma(2\gamma+n)}{\Gamma(\beta)\Gamma(2\gamma+n-\beta)} \int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma+n-\beta-1}\, du \tag{27}$$

This is a well known integral representation of the ${}_1F_1$-function. Then

$$\frac{\Gamma(2\gamma)}{\Gamma(\beta)\Gamma(2\gamma-\beta)} \int_0^1 e^{xu} u^{\beta-1} (1-u)^{2\gamma-\beta-1} {}_1F_1(\beta'; 2\gamma-\beta; y(1-u))\, du =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta')_n y^n}{(2\gamma)_n\, n!} {}_1F_1(\beta; 2\gamma+n; x) =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta')_n (\beta)_m y^n x^m}{(2\gamma)_{m+n}\, m!\, n!} = \Phi_2(\beta, \beta'; 2\gamma; x, y).$$

For the choice $\beta + \beta' = 2\gamma$ we obtain

$$\Phi_2(\beta, \beta'; \beta+\beta'; x, y) = \frac{\Gamma(2\gamma)}{\Gamma(\beta)\Gamma(\beta')} e^y \int_0^1 e^{(x-y)u} u^{\beta-1} (1-u)^{\beta'-1}\, du \tag{28}$$

since
$$ {}_1F_1(\beta'; \beta'; y(1-u)) = e^{y(1-u)}.$$

This shows, as we already know, that the Φ_2-function with the indicated relation between the parameters is of the type

$$\Phi_2(\beta, \beta'; \beta+\beta'; x, y) = e^y f(x-y)$$

Using (27) we obtain

$$\Phi_2 \beta, \beta' \beta+\beta'; x, y) = e^y {}_1F_1(\beta; \beta+\beta'; x-y) \tag{29}$$

and for the parameters (8)

$$\Phi_2(\gamma+i\eta, \gamma-i\eta; 2\gamma; ipr, -ipr) = e^{-ipr} {}_1F_1(\gamma+i\eta; 2\gamma; 2ipr)$$

in agreement with the previous result. The interesting result also previously derived

$$\Phi_2(\beta, \beta'; \beta+\beta'; x, y) = e^{-\lambda} \Phi_2(\beta, \beta'; \beta+\beta'; x+\lambda, y+\lambda) \tag{30}$$

is an easy consequence of the integral representation and is an example of an useful transformation, in that it allows a certain arbitrariness in the choice of the arguments of the function which is of importance in integrals over Coulomb wave functions where this freedom may be used to select convergent expansions.

Expansions of the wave functions

Besides the power series expansion, the general term of which is derived in (11), a number of expansions of the wave functions in terms of hypergeometric functions may be found. The term hypergeometric function is here to be interpreted in its more general sense. Expansions of this kind may be obtained by expanding the Laplace transform of the wave function and then apply the inverse Laplace transform to the terms.

We have, integrating term by term,

$$L\{t^{2\gamma-1}\Phi_2(\beta, \beta'; 2\gamma; xt, yt)\} = \int_0^\infty e^{-st} t^{2\gamma-1} \Phi_2(\beta, \beta'; 2\gamma; xt, yt)\, dt =$$

$$= \frac{\Gamma(2\gamma)}{s^{2\gamma}} \left(1-\frac{x}{s}\right)^{-\beta} \left(1-\frac{y}{s}\right)^{-\beta'} \tag{31}$$

Expanding

$$e^{\frac{z}{s}} \left(1-\frac{x}{s}\right)^{-\beta} \left(1-\frac{y}{s}\right)^{-\beta'} = \sum_{n=0}^{\infty} a_n \frac{1}{s^n}$$

where z is an arbitrary parameter, the Laplace transform can be written

$$L\{t^{2\gamma-1}\Phi_2(\beta, \beta'; 2\gamma; xt, yt)\} = \Gamma(2\gamma) \sum_{n=0}^{\infty} a_n \frac{e^{-\frac{z}{s}}}{s^{2\gamma+n}}\cdot \tag{32}$$

But

$$L\{t^{2\gamma+n-1} {}_0F_1(2\gamma+n; -zt)\} = \frac{\Gamma(2\gamma+n)}{s^{2\gamma+n}} e^{-\frac{z}{s}} \tag{33}$$

where the definition of the ${}_0F_1$-function is as in (13). This function is essentially a Bessel function. The relation is

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1; -z^2/_4\right).$$

From (32) and (33) it follows that

$$L\{t^{2\gamma-1}\Phi_2(\beta,\beta';2\gamma;xt,yt)\} = \sum_{n=0}^{\infty} L\left\{\frac{a_n}{(2\gamma)_n}\,t^{2\gamma+n-1}\,{}_0F_1(2\gamma+n;-zt)\right\}$$

from which we conclude

$$\Phi_2(\beta,\beta';2\gamma;xt,yt) = \sum_{n=0}^{\infty}\frac{a_n}{(2\gamma)_n}\,t^n\,{}_0F_1(2\gamma+n;-zt) \tag{34}$$

when the right hand side is uniformly convergent.

From (32) we obtain

$$a_n = \sum_{k+l+m=n}\frac{(\beta)_k(\beta')_l\,x^k y^l z^m}{k!\,l!\,m!}$$

which we shall prove to be equal to

$$a_n = \frac{(\beta')_n y^n}{n!}\Phi_1\left(-n,\beta;1-\beta'-n;\frac{x}{y},\frac{z}{y}\right) \tag{35}$$

where the Φ_1-function is one of the confluent Appell functions. It is defined by

$$\Phi_1(a,b;c;x,y) = \sum_{\substack{m=0\\n=0}}^{\infty}\frac{(a)_{m+n}(b)_m x^m y^n}{(c)_{m+n}\,m!\,n!}. \tag{36}$$

Using this definition in (35) we obtain

$$\frac{(\beta')_n y^n}{n!}\Phi_1\left(-n,\beta;1-\beta'-n;\frac{x}{y},\frac{z}{y}\right) = \frac{(\beta')_n y^n}{n!}\sum_{k,m}\frac{(-n)_{k+m}(\beta)_k\left(\frac{x}{y}\right)^k\left(\frac{z}{y}\right)^m}{(1-\beta'-n)_{k+m}\,k!\,m!} =$$

$$= \sum_{k,m}\frac{(-n)_{k+m}(\beta')_n(\beta)_k x^k y^{n-k-m} z^m}{n!\,(1-\beta'-n)_{k+m}\,k!\,m!}.$$

But
$$\frac{(-n)_{k+m}}{n!} = \frac{(-n)_{n-l}}{n!} = \frac{(-1)^{l+n}}{l!}$$

and

$$\frac{(\beta')_n}{(1-\beta'-n)_{k+m}} = \frac{(\beta')_n}{(1-\beta'-n)_{n-l}} = \frac{\Gamma(\beta'+n)\Gamma(1-\beta'-n)}{\Gamma(\beta')\Gamma(1-\beta'-l)} =$$

$$= \frac{\Gamma(\beta'+l)}{\Gamma(\beta')}\,\frac{\sin\pi(\beta'+l)}{\sin\pi(\beta'+n)} = (-1)^{l+n}(\beta')_l.$$

Then $$\frac{(\beta')_n y^n}{n!}\,\Phi_1\left(-n,\beta;1-\beta'-n;\frac{x}{y},\frac{z}{y}\right)=\sum_{k+l+m=n}\frac{(\beta)_k(\beta')_l\,x^k y^l z^m}{k!\,l!\,m!}$$

and (35) is proved.

An interesting feature of the expansion (34) is that the dependence of the energy E can be separated from the function depending on r and thus be contained in the coefficients only. In problems where a knowledge of the wave functions in the neighbourhood of the origin is needed for not too high energies the expansion should be quite useful.

For $\beta+\beta'=2\gamma$ the left hand side of (34) is an arbitrary exponential function times a radial Coulomb wave function, and if also $x=-y=\frac{ip}{2\alpha Z}$ the function $\Phi(\gamma,E,t)$ is obtained:

$$\Phi(\gamma,E,t)=\sum\frac{a_n}{(2\gamma)_n}t^n\,{}_0F_1(2\gamma+n;-zt) \tag{37}$$

where $$a_n=\frac{(\beta)_n\left(\frac{ip}{2\alpha Z}\right)^n}{n!}\,\Phi_1\left(-n,\beta';1-\beta-n;-1;\frac{z}{\frac{ip}{2\alpha Z}}\right). \tag{38}$$

Expansions of confluent hypergeometric functions have been given by F. Tricomi [7] and one of his expansions is obtained from the one given here (37) for $z=E$. Tricomi gives a recursion formula for the coefficients in this case which in our notation is

$$(n+1)\,a_{n+1}=-\frac{p^2}{4\alpha^2Z^2}\left[(n+2\gamma-1)\,a_{n-1}-E\,a_{n-2}\right] \tag{39}$$

$$a_0=1,\quad a_1=0,\quad a_2=-\frac{\gamma p^2}{4\alpha^2Z^2}$$

$$a_3=\frac{Ep^2}{12\alpha^2Z^2},\quad a_4=\frac{(\gamma)_2\,p^4}{32\alpha^4Z^4},\ \text{etc.}$$

The coefficient a_n is a polynomial in p with possible terms

$$a_n=c_n\,p^n+c_{n-1}\,p^{n-1}\,\dots\,+c_{n-\left[\frac{n}{3}\right]}\,p^{n-\left[\frac{n}{3}\right]} \tag{40}$$

where $\left[\frac{n}{3}\right]$ means the largest integer $\leq\frac{n}{3}$, but no odd powers occur. As an example, the coefficient a_{15} is of the form

$$a_{15}=c_{10}\,p^{10}+c_{12}\,p^{12}+c_{14}\,p^{14}$$

according to (40) but with all odd powers of p omitted.

The series (34) contains (7) as a special case for $z=0$ since then

$$a_n=\frac{(\beta')_n y^n}{n!}\Phi_1\left(-n,\beta;1-\beta'-n;\frac{x}{y},0\right)=\frac{(\beta')_n y^n}{n!}\,{}_2F_1\left(-n,\beta;1-\beta'-n;\frac{x}{y}\right)$$

Using a well known transformation of the ${}_2F_1$-function we obtain

$$a_n=\frac{y^n(\beta+\beta')_n}{n!}\,{}_2F_1\left(-n,\beta;\beta+\beta';1-\frac{x}{y}\right)$$

which gives the expansion (7). The general expansion (34) thus contains the power series expansion as well as different expansions in cylindrical functions and has been derived mainly in order to study the energy dependence of matrix elements for transitions between Coulomb states.

A direct use of the Tricomi expansion, e.g. in the matrix elements for electromagnetic transitions from a bound to a non-bound state, leads to an expansion that converges only for very small photon energies. The more general expansion given here and particularly the use of the Appell functions greatly increases the number of representations of the Coulomb matrix elements. This may lead to more convenient ways of computing them. As an example the radiative transition matrix element from a bound to a non-bound state as occurring in internal conversion can be expressed by means of the Appell function F_1 and the series representation of this function will then converge for all photon energies. The double series expansion can be expressed as a single series. For high photon energies the convergence is slow but tentative numerical investigations indicate that the expansion is practically useful for photon energies as high as ten times the binding energy of the emitted electron.

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REFERENCES

[1] P. Appell, J. Kampé de Fériet. Fonctions Hypergéométriques et Hypersphériques. Gauthier-Villars, Paris 1926.
A. Erdélyi *et al.* Higher Transcendental Functions Vol. I, p. 222. McGraw-Hill Book Company Inc. New York 1953.
[2] K. Alder, A. Bohr, T. Huus, B. Mottelson and A. Winther. Rev. Mod. Phys. 432, 28, 1956.
[3] H. A. Bethe and E. E. Salpeter. Handbuch der Physik, Vol. XXXV, p. 149. Springer Verlag, Berlin, Göttingen, Heidelberg 1957.
M. E. Rose, Phys. Rev. 51, 484, 1937.
[4] A. Erdélyi *et al.* Higher Transcendental Functions, Vol. III, p. 267. McGraw-Hill Book Company Inc. New York 1955.
[5] M. E. Rose. Phys. Rev. 51, 484, 1937.
[6] A. Erdélyi, Proc. Roy. Soc., Edinburgh, LIX, p. 224, 1939.
[7] A. Erdélyi *et al.* Higher Transcendental Functions, Vol. I, p. 277, McGraw-Hill Book Company Inc. New York 1953.

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The diffusion cooling effect in heavy water

By N. G. SJÖSTRAND[1]

An attempt has been made to determine the diffusion parameters of heavy water using the pulsed neutron source method [1]. Because of the rather low counting rates no complete investigation was possible. However, some results were obtained for the diffusion cooling coefficient and they are reported here.

Two quantities of heavy water were used: 3.68 and 3.14 kg with an H_2O impurity of 0.60 ± 0.10 and 0.75 ± 0.10 per cent by weight respectively. The heavy water was contained in a cylinder 18 cm in diameter, and the experimental arrangement was essentially the same as in earlier work [2]. The temperature of the moderator was 20°C. Measurements were made with different pulse lengths and channel widths and with different detector positions.

The bucklings for the two geometries were calculated as 0.0864 and 0.0966 cm^{-2}. The uncertainty is about 0.002 cm^{-2}, and it arises from the estimation of the extrapolation length. For pure D_2O at 20°C the transport mean free path was assumed to be 2.51 ± 0.04 cm [3, 4, 5]. The value adopted for the extrapolation length was 1.7 ± 0.1 cm, taking the impurity and decreased neutron temperature into account.

The measured time constants for the two bucklings were $14\,640 \pm 200$ and $16\,840 \pm 400$ sec^{-1}. Subtracting the small contribution from absorption, dividing by the bucklings and correcting for the H_2O impurity, the diffusion constants for pure D_2O are 1.74 ± 0.06 and 1.80 ± 0.08, both in units of 10^5 cm^2/sec. The value related to the smaller buckling is smaller than the other, but considering the limits of error this is not unreasonable.

In order to calculate the diffusion cooling coefficient c we use the relation

$$D = D_0 - (c - d) B^2,$$

where D_0 is the diffusion constant at zero buckling and d is a correction term [6] which for heavy water is estimated to 1.3×10^5 cm^4/sec. Using a value of (2.08 ± 0.03) 10^5 cm^2/sec for D_0 [3, 4, 5], $c - d$ is found to be (3.5 ± 0.8) 10^5 cm^4/sec, and the diffusion cooling coefficient is (4.8 ± 1.0) 10^5 cm^4/sec. This may be compared with the theoretical value 8.3×10^5 cm^4/sec from the assumption of a monatomic gas [7] with nuclei of mass 20. The discrepancy indicates that the average energy transfer between neutrons and moderator is somewhat larger than the theory predicts.

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REFERENCES

1. VON DARDEL, G., and SJÖSTRAND, N. G., Progr. Nucl. En. Ser. 1, Vol. 2, p. 183.
2. VON DARDEL, G., and SJÖSTRAND, N. G., Phys. Rev. *96*, 1245 (1954).
3. DEXTER, A. H., HAMERMESH, B., HONES, E. W., MORRIS, P. A., and RINGO, G. R., ANL-4746 (1951).
4. KASH, S. W., and WOODS, D. C., Phys. Rev. *90*, 564 (1953).
5. RAIEVSKI, V., and HOROWITZ, J., Comptes Rendus *238*, 1993 (1954).
6. SJÖSTRAND, N. G., *15*, 147 (1959).
7. VON DARDEL, G., Trans. Roy. Inst. Technol. *75* (1954).

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On the theory underlying diffusion measurements with pulsed neutron sources

By N. G. SJÖSTRAND

SUMMARY

Starting from the time-dependent Boltzmann equation some problems relating to measurements with pulsed neutron sources in small geometries are investigated. It is shown that the commonly used formula for the time constant of the fundamental mode of the neutron distribution is only approximate. As a consequence all diffusion cooling coefficients measured up to now are too small by 5 to 30 per cent. Further, the conditions under which the results from pulsed neutron source measurements in small geometries can be applied to a reactor are given, and the boundary conditions to use in the interpretation of such experiments are discussed.

Introduction

During the last few years many investigations of the neutron diffusion properties of moderators have been made with the pulsed neutron source method in a small geometry [1–10]. The time decay constant λ_0 of the fundamental mode of the neutron distribution is then given by

$$\lambda_0 = \lambda_a + DB_g^2, \tag{1}$$

where λ_a is the absorption probability per unit time, D the diffusion constant for neutron density and B_g^2 the geometric buckling. As the neutron temperature varies with buckling due to the diffusion cooling effect and D is temperature dependent, we can write in the first approximation

$$D = D_0 - CB_g^2, \tag{2}$$

where C is the diffusion cooling coefficient.

In the experiments the time constant λ_0 is measured as a function of the buckling. By fitting the data to the parabola

$$\lambda_0 = \lambda_a + D_0 B_g^2 - CB_g^4 \tag{3}$$

it is possible to determine the parameters λ_a, D_0 and C. (For details of the method see reference 11). Conversely, if the parameters are known for a moderator, eq. (3) can be used for determination of the geometric buckling. Such measurements can give useful information for reactors of peculiar shapes, on the influence of control rods etc. [6, 9].

Eq. (1) is based on the ordinary diffusion approximation with all its limitations. In spite of the large experimental effort in this field, no thorough discussion of the theory underlying the pulsed source experiments has appeared in the literature before. In his book on neutron transport theory Davison [12] shows that the time-dependent Boltzmann equation can be reduced to the same form as the stationary equation, and he thereafter only treats the stationary case in detail. Even though many of the results obtained for the stationary problems can thus be applied directly to the time-dependent ones, the points of interest are so different in the two cases that it is worth-while to go into some detail.

The questions to be considered are the following:

1. Under which conditions can results obtained with pulsed source measurements in a small geometry be applied to a reactor?
2. When is equation (1) valid?
3. What are the correct boundary conditions to use?

Reference will constantly be made to the work of Davison [12] by the abbreviation D. followed by the section number in his book.

The equivalence between the stationary and time-dependent problems

As the discussion following will be mainly based on the equivalence between the time-dependent and stationary problems, we first recapitulate the basis of this equivalence (D. 3.2).

We consider a single, homogeneous, finite body, containing no sources, and assume also that no fissionable material is present. The Boltzmann equation then takes the form

$$\frac{\partial N(\boldsymbol{r}, v\boldsymbol{\Omega}, t)}{\partial t} + v\boldsymbol{\Omega}\,\mathrm{grad}\,N(\boldsymbol{r}, v\boldsymbol{\Omega}, t) + v\,\Sigma_{\mathrm{tot}}(v)\,N(\boldsymbol{r}, v\boldsymbol{\Omega}, t)$$
$$= \int v'\,\Sigma_s(v')\,dv' \iint N(\boldsymbol{r}, v'\,\boldsymbol{\Omega}', t)\,f(v'\,\boldsymbol{\Omega}'\to v\boldsymbol{\Omega})\,d\boldsymbol{\Omega}'. \quad (4)$$

Here $N(\boldsymbol{r}, v\boldsymbol{\Omega}, t)\,dV\,dv\,d\boldsymbol{\Omega}\,dt$ is the number of neutrons at time t in the volume element dV around the point $\boldsymbol{r}$, travelling with speed between v and $v+dv$ in a direction lying within a solid angle $d\boldsymbol{\Omega}$ around the direction $\boldsymbol{\Omega}$. With our assumption of no fissionable material the total cross section Σ_{tot} is given by $\Sigma_{\mathrm{tot}} = \Sigma_s + \Sigma_c$, where Σ_s is the scattering and Σ_c the capture cross section. The scattering function $f(v'\,\boldsymbol{\Omega}'\to v\boldsymbol{\Omega})$ must be normalized so that

$$\iiint f(v'\,\boldsymbol{\Omega}'\to v\boldsymbol{\Omega})\,dv\,d\boldsymbol{\Omega} = 1. \quad (5)$$

Equation (4) is valid for the neutrons in a moderator after the injection of a burst of neutrons. If we make a Laplace transformation of this equation with respect to time, we get an equation for $\bar{N}(\boldsymbol{r}, v\boldsymbol{\Omega}, \lambda)$, where

$$\bar{N}(\boldsymbol{r}, v\boldsymbol{\Omega}, \lambda) = \int_0^\infty N(\boldsymbol{r}, v\boldsymbol{\Omega}, t)\,e^{-\lambda t}\,dt. \quad (6$$

The resulting equation is identical with the stationary form of eq. (4), if th macroscopic capture cross section is increased by λ/v, and of course $N(\boldsymbol{r}, v\boldsymbol{\Omega}, t$ is replaced by $\bar{N}(\boldsymbol{r}, v\boldsymbol{\Omega}, \lambda)$.

As there is no source term in the equation, we have an eigenvalue problem, because $\bar{N}(\boldsymbol{r}, v\boldsymbol{\Omega}, \lambda)$ has to satisfy certain boundary conditions. Our assumption of a finite body makes the eigenvalues form a discrete set. The largest eigenvalue λ_0 is the time constant of the system, and as we have assumed that no fissile material is present, it is necessarily negative. It follows that the decay of the neutron population sufficiently long after injection is exponential with the time constant λ_0.

Once we know this, it is easy to verify the above result on the apparent increase (or rather decrease, as λ_0 is negative) of the capture cross section by the substitution

$$N(\boldsymbol{r}, v\boldsymbol{\Omega}, t) = N(\boldsymbol{r}, v\boldsymbol{\Omega})\, e^{\lambda_0 t} \tag{7}$$

in eq. (4).

In the experiments which are of interest here, fast neutrons are injected into a finite moderator. The neutrons slow down to thermal energies and reach an equilibrium state, characterized by a constant spatial and velocity distribution, apart from the exponential decrease of the amplitude with the time constant λ_0. It may be possible to find systems where this condition is not exactly fulfilled during times suitable for measurements. However, for most moderators the slowing down times are short compared to the life times of the thermal neutrons, and it is thus possible to await the thermal equilibrium and also to eliminate the higher harmonic modes. In fact, a pure exponential decay has sometimes been found experimentally over four decades. In the following we shall therefore only be concerned with the fundamental mode and its time constant λ_0.

The leakage and absorption rates

If the fictitious capture is included in the stationary Boltzmann equation this can be written

$$\begin{aligned} v\boldsymbol{\Omega} \operatorname{grad} N(\boldsymbol{r}, v\boldsymbol{\Omega}) + [-\lambda_0 + v(\Sigma_c(v) + \Sigma_s(v))]\, N(\boldsymbol{r}, v\boldsymbol{\Omega}) \\ = \int v'\Sigma_s(v')\,dv' \iint N(\boldsymbol{r}, v'\boldsymbol{\Omega}')\, f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})\, d\boldsymbol{\Omega}', \end{aligned} \tag{8}$$

where we have changed the sign of λ_0 to get it positive. Integration over all neutron directions $\boldsymbol{\Omega}$, all velocities v, and over the whole volume V of the body gives

$$\lambda_0 = \frac{\iiiint\!\!\iint \Sigma_c(v)\, v\, N(\boldsymbol{r}, v\boldsymbol{\Omega})\, dv\, d\boldsymbol{\Omega}\, dV}{\iiiint\!\!\iint N(\boldsymbol{r}, v\boldsymbol{\Omega})\, dv\, d\boldsymbol{\Omega}\, dV} + \frac{\iiiint\!\!\iint v\boldsymbol{\Omega} \operatorname{grad} N(\boldsymbol{r}, v\boldsymbol{\Omega})\, dv\, d\boldsymbol{\Omega}\, dV}{\iiiint\!\!\iint N(\boldsymbol{r}, v\boldsymbol{\Omega})\, dv\, d\boldsymbol{\Omega}\, dV}. \tag{9}$$

This follows because the integral of the right side of eq. (8) over v and $\boldsymbol{\Omega}$ is equal to

$$\iiint v\,\Sigma_s(v)\, N(\boldsymbol{r}, v\boldsymbol{\Omega})\, dv\, d\boldsymbol{\Omega}$$

according to eq. (5).

The first term in eq. (9) is the absorption probability per unit time, which we have denoted by λ_a in eq. (1). If the absorption cross section Σ_c follows the $1/v$ law, then $\lambda_a = v \cdot \Sigma_c(v)$ independent of the neutron spectrum. This fundamental result has been used for cross section measurements [13]. If decay curves are measured with and without an extra absorber and the results are divided point by point, one gets an exponential with the time constant $v \cdot \Sigma_c'(v)$ where Σ_c' is the capture cross section of the extra absorber. As the spatial neutron distribution generally is very little changed by the addition of an absorber this exponential is obtained also in the presence of higher harmonic modes. Thus apart from very small corrections for the change in the spatial distribution of the neutrons due to the added absorber and for finite resolution in the measurement, the exponential behaviour is quite general even for rather short times after the neutron burst.

If the capture cross section does not follow the $1/v$ law the first term in eq. (9) will depend on the neutron spectrum. This fact has been used to study the effective neutron temperature as a function of the buckling of the geometry [14].

In the second term of eq. (9) $v\boldsymbol{\Omega}$ grad $N = \operatorname{div}(v\boldsymbol{\Omega} N)$. Then Gauss' theorem shows that the numerator is the net outflow of neutrons through the surface of the body. This term represents thus the leakage probability per unit time, and corresponds to the second term on the right side of eq. (1).

The equivalence between a reactor and a pulsed moderator

We assume now and in the following that the capture cross section follows the $1/v$ law. The time constant is then

$$\lambda_0 = v\,\Sigma_c + \gamma, \qquad (10)$$

where γ is the leakage rate. We introduce the quantities

$$l^*(v) = \frac{1}{\Sigma_s(v) - \gamma/v}, \qquad (11)$$

$$c^*(v) = \frac{\Sigma_s(v)}{\Sigma_s(v) - \gamma/v} \qquad (12)$$

and can then write equation (8) as follows.

$$v\boldsymbol{\Omega}\ \mathrm{grad}\ N(\boldsymbol{r}, v\boldsymbol{\Omega}) + \frac{v\,N(\boldsymbol{r}, v\boldsymbol{\Omega})}{l^*(v)} = \int \frac{v'\,c^*(v')\,dv'}{l^*(v')} \iint N(\boldsymbol{r}, v'\boldsymbol{\Omega}')\, f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})\, d\boldsymbol{\Omega}' \qquad (13$$

This equation is identical with that for the neutron density in a critical reactor if instead of the quantities $l^*(v)$ and $c^*(v)$ we use the ordinary total mean free path and number of secondaries per collision respectively. They are defined as

$$l(v) = \frac{1}{\Sigma_s(v) + \Sigma_c(v) + \Sigma_f(v)}, \qquad (14$$

$$c(v) = \frac{\Sigma_s(v) + \nu\,\Sigma_f(v)}{\Sigma_s(v) + \Sigma_c(v) + \Sigma_f(v)}, \qquad (15$$

where Σ_f is the fission cross section and ν the number of neutrons emitted per fission. Also the function $f(v'\mathbf{\Omega}' \rightarrow v\mathbf{\Omega})$ has a different form in the presence of fission neutrons (D. 2.2). Eq. (13) can be considered as an eigenvalue problem in $c(v)$ or $c^*(v)$ and we see that both for a critical reactor and a pulsed moderator c must be larger than unity.

The important conclusions from the above equivalence can be stated as follows: We have a bare, homogenous, critical reactor characterized by $c(v), l(v)$ and a scattering function and a pulsed moderator characterized by $c^*(v)$, $l^*(v)$ and a scattering function. The eigenvalues $c(v)$ and $c^*(v)$ are identical, if

(*a*) the functions $f(v'\mathbf{\Omega}' \rightarrow v\mathbf{\Omega})$ in the two systems are the same,

(*b*) $l(v)/l^*(v) = \text{const.}$,

(*c*) the ratio of the linear dimensions of the two systems is equal to this ratio between the total mean free paths.

This result, though very general, is of little practical importance, as conditions (*a*) and (*b*) are seldom fulfilled. We will therefore try to find similar conditions for the one-group approximation (called by Davison the constant cross section approximation).

Two assumptions are made. First, the neutron spectra in the systems are supposed to be independent of position. We can then write

$$N(\boldsymbol{r}, v\mathbf{\Omega}) = \psi(\boldsymbol{r}, \mathbf{\Omega})\, F(v). \tag{16}$$

Secondly, the angular distribution of neutrons resulting from collisions is supposed to be independent of the velocity of the neutrons. This means

$$f(v'\mathbf{\Omega}' \rightarrow v\mathbf{\Omega}) = g(v' \rightarrow v)\, f(\mathbf{\Omega}' \rightarrow \mathbf{\Omega}). \tag{17}$$

Making these substitutions in eq. (13) and integrating over all v, we obtain (D. 4.4)

$$\mathbf{\Omega} \operatorname{grad} \psi(\boldsymbol{r}, \mathbf{\Omega}) \int_0^\infty v\, F(v)\, dv + \psi(\boldsymbol{r}, \mathbf{\Omega}) \int_0^\infty \frac{v\, F(v)}{l(v)}\, dv$$
$$= \int\!\!\int \psi(\boldsymbol{r}, \mathbf{\Omega}')\, f(\mathbf{\Omega}' \rightarrow \mathbf{\Omega})\, d\mathbf{\Omega}' \int_0^\infty \frac{v'\, c(v')\, F(v')}{l(v')}\, dv', \tag{18}$$

where we have omitted the asterisks. We may define averages l and c of $l(v)$ and $c(v)$ by means of

$$\frac{1}{l} \int_0^\infty v\, F(v)\, dv = \int_0^\infty \frac{v\, F(v)\, dv}{l(v)}, \tag{19}$$

$$c \int_0^\infty \frac{v\, F(v)\, dv}{l(v)} = \int_0^\infty \frac{v\, c(v)\, F(v)\, dv}{l(v)}. \tag{20}$$

Thus eq. (18) is finally reduced to

$$\mathbf{\Omega} \operatorname{grad} \psi(\boldsymbol{r}, \mathbf{\Omega}) + \frac{\psi(\boldsymbol{r}, \mathbf{\Omega})}{l} = \frac{c}{l} \int\!\!\int \psi(\boldsymbol{r}, \mathbf{\Omega}')\, f(\mathbf{\Omega}' \rightarrow \mathbf{\Omega})\, d\mathbf{\Omega}'. \tag{21}$$

This equation and the definitions of the averages are the same as Davison found but are more generally valid as we have not assumed the scattering to be isotropic.

If the definitions (11) and (12) are used in eqs. (19) and (20) we can express l^* and c^* in terms of the average scattering cross section. We choose to normalize the neutron *density* spectrum, that is

$$\int_0^\infty F(v)\,dv = 1. \tag{22}$$

With the average scattering cross section defined as

$$\bar{\Sigma}_s = \frac{1}{\bar{v}} \int_0^\infty v\,F(v)\,\Sigma_s(v)\,dv \tag{23}$$

we find then

$$l^* = \frac{1}{\bar{\Sigma}_s - \gamma/\bar{v}}, \tag{24}$$

$$c^* = \frac{\bar{\Sigma}_s}{\bar{\Sigma}_s - \gamma/\bar{v}}, \tag{25}$$

where $\bar{v}$ is the average neutron velocity. For a critical reactor the expressions (14) and (15) are unchanged when the averages are introduced instead of the velocity dependent quantities.

If c is close to unity the effect of anisotropic scattering on the solution of eq. (21) is in first approximation equivalent to changing the total mean free path to the transport mean free path (D. 17.3). We can then formulate the conditions for getting the same eigenvalues c and c^*, namely

(*a*) the neutron spectra in the two systems must be independent of position,

(*b*) the angular distribution of neutrons resulting from collisions must be independent of velocity,

(*c*) the eigenvalues c and c^* must be close to unity,

(*d*) the linear dimensions of the systems must be the same when measured in units of the respective transport mean free paths.

The equivalence between diffusion length and buckling

We are now going to study the solution of equation (21), and we start with assuming that the scattering is isotropic. It has been shown (D. 5.1) that for an infinite medium a solution to the corresponding integral equation can then be obtained if

$$\frac{1}{c} = \frac{L}{2l} \log \frac{L+l}{L-l} \tag{26}$$

and the solution satisfies the equation

$$\nabla^2 \varrho(\boldsymbol{r}) = \varrho(\boldsymbol{r})/L^2, \tag{27}$$

where
$$\varrho(\boldsymbol{r}) = \iint \psi(\boldsymbol{r}, \boldsymbol{\Omega})\, d\boldsymbol{\Omega}. \tag{28}$$

The diffusion length L is a real quantity if $c < 1$. If $c > 1$, L becomes imaginary, but the derivation given by Davison is obviously valid also in this case. Introducing

$$B = \frac{1}{iL}, \tag{29}$$

we find the relation
$$\frac{1}{c} = \frac{1}{Bl} \tan^{-1} Bl \tag{30}$$

and the solution of the Boltzmann equation satisfies then the equation

$$\nabla^2 \varrho(\boldsymbol{r}) + B^2 \varrho(\boldsymbol{r}) = 0. \tag{31}$$

Even if we have derived these formulae with the assumption of an infinite medium we can also apply them approximately to finite systems if suitable boundary conditions are imposed. However, the flux $\varrho(\boldsymbol{r})$ calculated from this diffusion approximation will as usual be decidedly inaccurate within a boundary layer several mean free paths in thickness.

Obviously the above relation between B, l and c (eq. 30) is equally valid for both physical cases for which $c > 1$, namely for a critical reactor and for a pulsed moderator, if only we use the proper values for l and c. In both cases eq. (31) gives the neutron distribution, and the lowest eigenvalue determined by the boundary conditions is called the geometric buckling, B_g^2.

Eq. (30) can be interpreted for a reactor as giving the material buckling B_m^2 for a medium with the properties defined by c and l. For a critical reactor $B_m^2 = B_g^2$. This condition is already inherent in eqs. (30) and (31) and the boundary conditions because eq. (21) describes a source free, stationary state, i.e. criticality.

The relation between time constant and buckling for isotropic scattering

From eq. (30) we can now find a relation between the time constant and geometric buckling for a small moderator. By substituting l^* and c^* from eqs. (24) and (25) we get

$$\tan \frac{B_g}{\overline{\Sigma}_s} = \frac{B_g}{\overline{\Sigma}_s - \gamma/\bar{v}} \tag{32}$$

The leakage rate is thus

$$\gamma = \bar{v}\,\overline{\Sigma}_s - \frac{\bar{v}\, B_g}{\tan\,(B_g/\overline{\Sigma}_s)} \tag{33}$$

and the time constant

$$\lambda_0 = v\,\Sigma_c + \bar{v}\,\overline{\Sigma}_s \left(1 - \frac{B_g/\overline{\Sigma}_s}{\tan\; B_g/\overline{\Sigma}_s}\right). \tag{34}$$

This is a more exact formula than that given in eq. (1). As the ratio B_g/Σ_s is a small quantity in pulsed source experiments, we can make a series development and find

$$\lambda_0 = v\,\Sigma_c + \frac{\bar{v}\,B_g^2}{3\,\overline{\Sigma}_s}\left(1 + \frac{B_g^2}{15\,\overline{\Sigma}_s^2} + \frac{2\,B_g^4}{315\,\overline{\Sigma}_s^4} + \cdots\right). \tag{35}$$

This series corresponds to that obtained for L when the absorption is not small (D. 5.2). Usually for a moderator only the first term in the L-series has to be taken into account. However, for a pulsed source measurement the term B_g^4 is also important as the measured time constant is fitted to a parabola in B_g^2 and the coefficient for the B_g^4 term is usually interpreted as the diffusion cooling coefficient according to eq. (3). The order of magnitude of the B_g^6 term also may be of interest in assessing the validity of the assumed parabolic dependence.

Thus eq. (1) for the time constant should also contain a term $d\,B_g^4$ and the coefficient for B_g^4 in eq. (3) is then C-d. This means that all the diffusion cooling coefficients measured up to now should be increased by d.[1] Before calculating the magnitude of this correction we have to study the influence of anisotropic scattering.

The relation between time constant and buckling for anisotropic scattering

When the scattering is anisotropic it is convenient to express the scattering function in eq. (21) as an expansion in Legendre polynomials (D. 17.1)

$$f(\mathbf{\Omega}' \to \mathbf{\Omega}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1)\, b_n\, P_n(\mathbf{\Omega}\cdot\mathbf{\Omega}'), \tag{36}$$

where $b_0 = 1$. For simplicity we study here only the case of plane geometry and the Boltzmann equation then takes the form

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \frac{\psi(x,\mu)}{l} = \frac{c}{4\pi l} \sum_{n=0}^{\infty} (2n+1)\, b_n\, \psi_n(x)\, P_n(\mu), \tag{37}$$

where
$$\psi(x,\mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1)\, \psi_n(x)\, P_n(\mu) \tag{38}$$

and μ is the x-component of $\mathbf{\Omega}$.

Just as for isotropic scattering a relation between c, L and l can be derived (D. 17.3) but in the form of a determinantal equation. In the first order approximation it is found that if the transport cross section defined by

$$\Sigma_{\text{tr}} = \overline{\Sigma}_s (1 - b_1) \tag{39}$$

is introduced instead of the scattering cross section, the same expression for the diffusion length can be used. We can thus expect the expression for the time constant to be correct in the B_g^2 term if the transport cross section is introduced.

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[1] Kazarnovsky, Stepanov and Shapiro have mentioned this correction for the case of the P_3 approximation in plane geometry (Second Conf. on the Peaceful Uses of Atomic Energy, Paper P/2148, Geneva 1958).

However, due to the importance of the B_g^4 term for the interpretation of the diffusion cooling effect a higher order approximation is needed.

As the calculation and solution of the determinantal equation is rather laborious, the spherical harmonics method was used instead. Multiplication of eq. (37) with $(2n+1)P_n(\mu)$ and integration over all μ gives the following system of differential equations

$$(n+1)\psi'_{n+1}(x) + n\,\psi'_{n-1}(x) + (2n+1)\frac{1-c\,b_n}{l}\psi_n(x) = 0 \quad (n=0, 1, 2\ldots). \tag{40}$$

We seek solutions of the form

$$\psi_n(x) = g_n\,e^{i\alpha x} \tag{41}$$

and find that the following, simpler determinant must vanish:

$$\begin{vmatrix} 1-c & i\alpha l & 0 & 0 & \ldots \\ i\alpha l & 3(1-c\,b_1) & 2\,i\alpha l & 0 & \ldots \\ 0 & 2\,i\alpha l & 5(1-c\,b_2) & 3\,i\alpha l & \ldots \\ 0 & 0 & 3\,i\alpha l & 7(1-c\,b_3) & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \tag{42}$$

Substituting l^* and c^* from eqs. (24) and (25) and identifying the positive value for α^2 with B_g^2 gives the following result in the P_3 approximation

$$\lambda_0 = v\,\Sigma_c + \frac{\bar{v}\,B_g^2}{3\,\Sigma_{\mathrm{tr}}}\left(1 + \frac{B_g^2}{15\,\Sigma_{\mathrm{tr}}^2}\cdot\frac{1+4\,b_1-5\,b_2}{1-b_2}\right). \tag{43}$$

The P_5 approximation leads to the same expression if b_4 and b_5 are assumed negligible.

When the spherical harmonics method is applied to a spherically symmetrical system the relation between the time constant and the buckling is given by the same determinant (D. 11.1). Therefore eq. (43) can be regarded as a good approximation for the geometries used in pulsed source experiments.

As the B_g^6 term turns out to be very small in eq. (35) compared to the uncertainties in the dependence of the diffusion constant on the buckling it is not necessary to calculate the influence of the anisotropy on that term.

The corrections to the diffusion cooling coefficient

We can now estimate corrections to be applied to diffusion cooling coefficients which have been calculated from measurements by use of eq. (3). The corrections are given by the last term in eq. (43) which, we may recall, was derived under the assumption that the angular distribution in the scattering process is independent of neutron energy.

It is well known from theory and experiment that this assumption is not valid for most moderators, so the coefficients b_1 and b_2 have to be interpreted as some average values over the neutron spectrum. In lack of detailed information we assume that $b_2 = 0$, and the correction d, which is always positive, is then calculated from the formula

$$d = \frac{D_0 l_{tr}^2}{15}(1 + 4 b_1). \tag{44}$$

The averages used for b_1 of H_2O, CH_2 and D_2O have been obtained from eq. (39) where $\bar{\Sigma}_s$ has been calculated from measured curves and Σ_{tr} from measured values of l_{tr} taken from the literature. For H_2O and D_2O the curves given in BNL-325 were used, but no curve was available for CH_2. Therefore the cross section of polythene was measured using the slow and fast choppers at the Swedish R1 reactor.[1] For the moderators Be, BeO and graphite we made the reasonable assumption $b_1 = 0$.

The calculated corrections, d, are listed in table 1 together with adopted values for the parameters used and for the coefficient $C - d$ of the B_g^4 term in eq. (3). The errors given do not include the possible uncertainty from higher terms in eq. (2). As so many approximations have been made in the evaluation the result must be considered only to give an estimate of the magnitude, especially for the three first moderators. However, it is obvious that the corrections are by no means negligible.

Table 1.

Moderator	D_0 10^4 cm^2/sec	l_{tr} cm	b_1	d 10^5 cm^4/sec	$C-d$ 10^5 cm^4/sec	References for $C-d$
H_2O	3.52	0.425	0.25	0.0085	0.036 ± 0.015	[1, 2, 4, 5, 8]
CH_2	2.66	0.323	0.25	0.0037	0.012 ± 0.005	[8, 9]
D_2O	20.8	2.51	0.12	1.3	3.5 ± 0.8	[10]
Be	12.2	1.48	0	0.18	3 ± 1	[2, 5]
BeO	11.8	1.43	0	0.16	4 ± 1	[3, 7]
Graphite	21.2	2.56	0	0.93	14 ± 2	[2, 6]

The boundary conditions

In the preceding sections we have obtained relations between the time constant and the buckling. The next problem is to investigate how the geometric buckling is calculated from the dimensions of a body. This usually involves the introduction of the *extrapolated endpoint* x_0 which is defined as the distance beyond the surface of the body at which the asymptotic neutron flux vanishes (D. 6.2). This is not the same as the linear extrapolation length, which is the ratio between the asymptotic neutron flux and its derivative at the surface, and is only equal to x_0 when $c = 1$.

Because of the small dimensions of the moderators used in pulsed neutron source experiments great care must be taken to find the right value of x_0. A large error here will cause a large error in the buckling of small systems and hence also in the result obtained for the diffusion cooling coefficient. For the stationary case formulae for x_0 have been given for anisotropic scattering and for various values of c (D. 17.4) but they are not directly applicable to the small geometries used in pulsed source experiments.

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[1] My thanks are due to Dr. K.-E. Larsson and Mr. E. Johansson for arranging these measurements.

Therefore, a calculation was made for an infinite slab of thickness $2a$ using the P_3 approximation of the spherical harmonics method. We start then from the system of equations (40) assuming the coefficients b_2 and above to be zero. For c and l the expressions (24) and (25) are used. With the origin in the centre of the slab the solution for $\psi_0(x)$ is found to be of the form

$$\psi_0(x) = A_1 \cos \alpha x + A_2 \cosh \beta x, \tag{45}$$

where using eq. (42) for c^* close to unity α and β can be expressed as

$$\alpha^2 l^{*2} = 3(c^* - 1)(1 - c^* b_1)(1 + \tfrac{4}{5}(c^* - 1)), \tag{46}$$

$$\beta^2 l^{*2} = \frac{35 - 28(c^* - 1)}{9}. \tag{47}$$

From the system of equations (40) similar expressions for ψ_1, ψ_2 and ψ_3 are easily obtained.

The most appropriate boundary conditions are those named after Marshak (D. 10.3)

$$\int_{-1}^{0} \psi(a, \mu)\, \mu\, d\mu = 0, \qquad \int_{-1}^{0} \psi(a, \mu)\, \mu^3\, d\mu = 0. \tag{48}$$

Introducing the series expansion of eq. (38) they can also be expressed as

$$\frac{\psi_0(a)}{2} - \psi_1(a) + \frac{5}{8}\psi_2(a) = 0, \tag{49}$$

$$\frac{\psi_0(a)}{4} - \frac{3}{5}\psi_1(a) + \frac{5}{8}\psi_2(a) - \frac{2}{5}\psi_3(a) = 0. \tag{50}$$

Using these equations and approximating, assuming c^* to be close to unity, an expression for $\tan \alpha a$ is derived.

The asymptotic neutron flux is represented by the cosine term in eq. (45). The extrapolated endpoint is therefore defined by

$$\alpha(a + x_0) = \frac{\pi}{2} \tag{51}$$

and α^2 is just the geometric buckling B_g^2. In the first approximation we find

$$\frac{x_0}{l_{\mathrm{tr}}} = 0.7051\,[1 - B_g^2 l_{\mathrm{tr}}^2 (0.0256 - 0.2825\, b_1 + 0.0893\, b_1^2)]. \tag{52}$$

The factor 0.7051 is close to the correct value 0.7104, which shows that the calculation is rather accurate.

For $b_1 = 0$, $\quad x_0/l_{\mathrm{tr}} = 0.7051\,(1 - 0.0256\, B_g^2 l_{\mathrm{tr}}^2)$
For $b_1 = 0.25$, $\quad x_0/l_{\mathrm{tr}} = 0.7051\,(1 + 0.0394\, B_g^2 l_{\mathrm{tr}}^2)$.

These figures show that for systems used in pulsed source experiments, where $B_g^2 l_{\mathrm{tr}}^2$ is usually smaller than 0.1, the resulting correction to the extrapolated endpoint is so small that it can be neglected. The same conclusion was also reached in a similar calculation for a sphere. This is in agreement with the numerical results mentioned in reference 1, which were obtained with the assumption of isotropic scattering.

The finite geometry may cause indirectly another uncertainty in the choice of a suitable extrapolated endpoint. The diffusion cooling effect causes a change in the neutron spectrum which as a first approximation can be described as a decrease in the temperature of the Maxwell spectrum of the neutrons [15]. The average scattering cross section defined in eq. (23) and hence the transport mean free path may therefore vary with buckling. For water the decrease in neutron temperature is of the order of 70° per unit buckling (measured in cm^{-2}) [1] [14]. As the transport mean free path is rather accurately proportional to the square root of the neutron temperature the extrapolated endpoint will be about 10 per cent smaller at a buckling of 0.8 cm^{-2}. Corrections are therefore necessary.

Conclusions

Even if some results have been obtained for the problems discussed here no full understanding has been reached. More experimental data on the scattering functions of moderators are needed and also further theoretical investigations. Especially, the possibility of a spatial dependence of the neutron velocity distribution should be studied, as such an effect might be of importance in the small systems treated here.

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REFERENCES

1. von Dardel G., and Sjöstrand N. G., Phys. Rev. *96*, 1245 (1954).
2. Antonov A. V., Isakoff A. I., Murin I. D., Neupocoyev B. A., Frank I. M., Shapiro F. L., and Shtranich I. V., Proc. Int. Conf. on Peaceful Uses of Atomic Energy, Geneva 1955, *5*, 3.
3. Ramanna R., Mani G. S., Iyengar P. K., Iyengar S. B. D., and Joshi B. V., J. Nucl. Energy *2*, 145 (1956).
4. Bracci A., and Coceva C., Nuovo Cim. *4*, Ser. 10, 59 (1956).
5. Campbell E. C., and Stelson P. H., ORNL 2076 (1956).
6. Beckurts K. H., Nucl. Sci. Eng. *2*, 516 (1957).
7. Iyengar S. B. D., Mani G. S., Ramanna R., and Umakanth N., Proc. Ind. Acad. Sci. *45*, 215 (1957).
8. Dio W. H., Nukleonik *1*, 13 (1958).
9. Sjöstrand N. G., Second Int. Conf. on Peaceful Uses of Atomic Energy. Paper 15/P/161 (1958).
10. Sjöstrand N. G., Arkiv f. Fys. *15*, 145 (1959).
11. von Dardel G., and Sjöstrand N. G., Progress in Nuclear Energy, Ser. 1, Vol. 2, 183 (1958).
12. Davison B., Neutron Transport Theory, Clarendon Press, Oxford, 1957.
13. von Dardel G., and Sjöstrand N. G., Phys. Rev. *96*, 1566 (1954).
14. Santandrea E., Toselli F., and Viano G., Second Int. Conf. on Peaceful Uses of Atomic Energy. Paper 15/P/1372 (1958).
15. Beckurts K. H., Zeitschr. Naturf. *12a*, 956 (1957).

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